Exercise 5C

1 $z = \frac{y}{x} \Rightarrow z = yx^{-1}$

$$\frac{dz}{dx} = -yx^{-2} + x^{-1}\frac{dy}{dx}$$

$$\frac{dz}{dx} + \frac{y}{x^2} = \frac{1}{x}\frac{dy}{dx}$$

$$\frac{dy}{dx} = x\frac{dz}{dx} + \frac{y}{x}$$

$$= x\frac{dz}{dx} + z$$

a $\frac{dy}{dx} = \frac{y}{x} + \frac{x}{y}$

The transformed equation is:

$$x\frac{dz}{dx} + z = z + \frac{1}{z}$$

$$x\frac{dz}{dx} = \frac{1}{z}$$

$$z\frac{dz}{dx} = \frac{1}{x}$$

$$\int z\,dz = \int \frac{1}{x}\,dx$$

$$\frac{1}{2}z^2 = \ln x + c$$

$$= \ln x + \ln B \quad \text{where } c = \ln B$$

$$\frac{1}{2}z^2 = \ln xB$$

$$z^2 = 2\ln xB$$

$$= \ln x^2B^2$$

$$= \ln Dx^2 \quad \text{where } D = B^2$$

Since $z = \frac{y}{x}$

$$\frac{y^2}{x^2} = \ln Dx^2$$

$$y^2 = x^2 \ln Dx^2$$

1 b $\frac{dy}{dx} = \frac{y}{x} + \frac{x^2}{y^2}$

The transformed equation is:

$x\frac{dz}{dx} + z = z + \frac{1}{z^2}$

$x\frac{dz}{dx} = \frac{1}{z^2}$

$z^2\frac{dz}{dx} = \frac{1}{x}$

$\int z^2 \, dz = \int \frac{1}{x} \, dx$

$\frac{1}{3}z^3 = \ln x + c$

Since $z = \frac{y}{x}$

$\frac{y^3}{x^3} = 3(\ln x + c)$

$y^3 = 3x^3(\ln x + c)$

or $y^3 = x^3 \text{In} Dx^3$

c $\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$

The transformed equation is:

$x\frac{dz}{dx} + z = z + z^2$

$x\frac{dz}{dx} = z^2$

$\frac{dz}{dx} = \frac{z^2}{x}$

$\int \frac{1}{z^2} \, dz = \int \frac{1}{x} \, dx$

$-\frac{1}{z} = \ln x + c$

Since $z = \frac{y}{x}$

$-\frac{x}{y} = \ln x + c$

$y = -\frac{x}{\ln x + c}$

1 d $\frac{dy}{dx} = \frac{x^3 + 4y^3}{3xy^2} = \frac{x^2}{3y^2} + \frac{4y}{3x}$

The transformed equation is:

$x\frac{dz}{dx} + z = \frac{1}{3z^2} + \frac{4}{3}z$

$x\frac{dz}{dx} = \frac{1}{3z^2} + \frac{1}{3}z$

$= \frac{1+z^3}{3z^2}$

$\int \frac{3z^2}{1+z^3}dz = \int \frac{1}{x}dx$

$\ln(1+z^3) = \ln x + c$

$1 + z^3 = e^{\ln x + c}$

$= e^{\ln x}e^c$

$= Ax$ where $A = e^c$

$z^3 = Ax - 1$

Since $z = \frac{y}{x}$

$\frac{y^3}{x^3} = Ax - 1$

$y^3 = x^3(Ax - 1)$

2 a $z = y^{-2}$

$\frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}$

$\frac{dy}{dx} = -\frac{1}{2}y^3\frac{dz}{dx}$

$\frac{dy}{dx} + \frac{1}{2}(\tan x)y = -(2\sec x)y^3$

The transformed equation is:

$-\frac{1}{2}y^3\frac{dz}{dx} + \frac{1}{2}(\tan x)y = (-2\sec x)y^3$

$-\frac{1}{2}\frac{dz}{dx} + \frac{1}{2}(\tan x)y^{-2} = -2\sec x$

$\frac{dz}{dx} - (\tan x)y^{-2} = 4\sec x$

Since $z = y^{-2}$

$\frac{dz}{dx} - (\tan x)z = 4\sec x$ as required

2 b $\frac{\mathrm{d}z}{\mathrm{d}x} - (\tan x)z = 4\sec x$

$\mathrm{e}^{\int -\tan x \mathrm{d}x} = \mathrm{e}^{-In|\sec x|} = \cos x$

$\cos x \frac{dz}{dx} - z\sin x = 4$

$z\cos x = 4x + c$

$z = \frac{4x+c}{\cos x}$

Since $z = \frac{1}{y^2}$

$\frac{1}{y^2} = \frac{4x+c}{\cos x}$

$y = \sqrt{\frac{\cos x}{4x+c}}$

3 a $z = x^{\frac{1}{2}}$

$\frac{\mathrm{d}}{\mathrm{d}t}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}}\frac{\mathrm{d}x}{\mathrm{d}t}$

Therefore:

$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{1}{2}x^{-\frac{1}{2}}\frac{\mathrm{d}x}{\mathrm{d}t}$

$\frac{\mathrm{d}x}{\mathrm{d}t} = 2x^{\frac{1}{2}}\frac{\mathrm{d}z}{\mathrm{d}t}$

$= 2z\frac{\mathrm{d}z}{\mathrm{d}t}$

The original equation is:

$\frac{\mathrm{d}x}{\mathrm{d}t} + t^2x = t^2x^{\frac{1}{2}}$

The transformed equation is:

$2z\frac{\mathrm{d}z}{\mathrm{d}t} + t^2z^2 = t^2z$

$\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{1}{2}t^2z = \frac{1}{2}t^2$ as required

3 b $\frac{\mathrm{d}z}{\mathrm{d}t}+\frac{1}{2}t^2z=\frac{1}{2}t^2$

$e^{\frac{1}{2}\int t^2\,\mathrm{d}t}=e^{\frac{1}{6}t^3}$

$e^{\frac{1}{6}t^3}\frac{dz}{dt}+\frac{1}{2}t^2e^{\frac{1}{6}t^3}z=\frac{1}{2}t^2e^{\frac{1}{6}t^3}$

$ze^{\frac{1}{6}t^3}=e^{\frac{1}{6}t^3}+c$

$z=1+ce^{-\frac{1}{6}t^3}$

Since $z=x^{\frac{1}{2}}$

$x^{\frac{1}{2}}= +ce^{-\frac{1}{6}t^3}$

$x=\left(1+ce^{-\frac{1}{6}t^3}\right)^2$

4 a $z=y^{-1}$

$\frac{\mathrm{d}z}{\mathrm{d}x}=-y^{-2}\frac{\mathrm{d}y}{\mathrm{d}x}$

$\frac{\mathrm{d}y}{\mathrm{d}x}=-y^2\frac{\mathrm{d}z}{\mathrm{d}x}$

$=-z^{-2}\frac{\mathrm{d}z}{\mathrm{d}x}$

The original equation is:

$\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{1}{x}y=\frac{(x+1)^3}{x}y^2$

The transformed equation is:

$-z^{-2}\frac{\mathrm{d}z}{\mathrm{d}x}-\frac{1}{x}\left(\frac{1}{z}\right)=\frac{(x+1)^3}{x}\left(\frac{1}{z^2}\right)$

$-\frac{\mathrm{d}z}{\mathrm{d}x}-\frac{z}{x}=\frac{(x+1)^3}{x}$

$\frac{\mathrm{d}z}{\mathrm{d}x}+\frac{z}{x}=-\frac{(x+1)^3}{x}$

4 b $\frac{dz}{dx}+\frac{z}{x}=-\frac{(x+1)^3}{x}$

$e^{\int \frac{1}{x}dx}=e^{\ln x}=x$

$x\frac{dz}{dx}+z=-(x+1)^3$

$zx=-\left[\frac{(x+1)^4}{4}+c\right]$

$=-\frac{(x+1)^4}{4}-c$

$=-\frac{(x+1)^4-k}{4}$

Since $z=\frac{1}{y}$

$\frac{1}{y}=-\frac{(x+1)^4-k}{4x}$

$y=-\frac{4x}{(x+1)^4-k}$

$=\frac{4x}{k-(x+1)^4}$

5 **a** $z = y^2$

$\frac{dz}{dx} = 2y\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{2y}\frac{dz}{dx}$

$= \frac{1}{2\sqrt{z}}\frac{dz}{dx}$

The original equation is:

$2(1+x^2)\frac{dy}{dx} + 2xy = \frac{1}{y}$

$\frac{dy}{dx} + \frac{xy}{1+x^2} = \frac{1}{2y(1+x^2)}$

The transformed equation is:

$\frac{1}{2\sqrt{z}}\frac{dz}{dx} + \frac{x\sqrt{z}}{1+x^2} = \frac{1}{2\sqrt{z}(1+x^2)}$

$\frac{dz}{dx} + \left(\frac{2x}{1+x^2}\right)z = \frac{1}{(1+x^2)}$ as required

b $e^{\int \frac{2x}{1+x^2}dx} = e^{\ln(1+x^2)} = 1+x^2$

$(1+x^2)\frac{dz}{dx} + 2xz = 1$

$z(1+x^2) = x + c$

$z = \frac{x+c}{1+x^2}$

Since $z = y^2$

$y^2 = \frac{x+c}{1+x^2}$

$y = \sqrt{\frac{x+c}{1+x^2}}$

c When $x = 0$, $y = 2$

$2 = \sqrt{\frac{0+c}{1+0^2}}$

$\sqrt{c} = 2$

$c = 4$

$y = \sqrt{\frac{x+4}{1+x^2}}$

6 $z = y^{-(n-1)} \Rightarrow y = z^{-\frac{1}{n-1}}$

$$\frac{dz}{dx} = -(n-1)y^{-(n-1)-1}\frac{dy}{dx}$$

$$= -(n-1)y^{-n+1-1}\frac{dy}{dx}$$

$$= -(n-1)y^{-n}\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{(n-1)}y^{n}\frac{dz}{dx}$$

$$y^{n} = \left[z^{-\frac{1}{(n-1)}}\right]^{n}$$

$$= z^{-\frac{n}{(n-1)}}$$

$$\frac{dy}{dx} = -\frac{1}{(n-1)}z^{-\frac{n}{n-1}}\frac{dz}{dx}$$

The original equation is:

$$\frac{dy}{dx} + P(x)y = Q(x)y^{n}$$

The transformed equation is:

$$-\frac{1}{(n-1)}z^{-\frac{n}{n-1}}\frac{dz}{dx} + z^{-\frac{1}{n-1}}P(x) = Q(x)z^{-\frac{n}{n-1}}$$

$$z^{-\frac{n}{n-1}}\frac{dz}{dx} - (n-1)P(x)z^{-\frac{1}{n-1}} = -(n-1)Q(x)z^{-\frac{n}{n-1}}$$

$$\frac{dz}{dx} - (n-1)P(x)z^{-\frac{1}{n-1}} \times z^{\frac{n}{n-1}} = -(n-1)Q(x)$$

$$\frac{dz}{dx} - (n-1)P(x)z^{\frac{n-1}{n-1}} = -(n-1)Q(x)$$

$$\frac{dz}{dx} - (n-1)P(x)z = -(n-1)Q(x) \quad \text{as required}$$

7 a $u = y + 2x \Rightarrow y = u - 2x$

$\frac{du}{dx} = \frac{dy}{dx} + 2 \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 2$

The original equation is:

$$\frac{dy}{dx} = -\frac{(1 + 2y + 4x)}{1 + y + 2x}$$

The transformed equation is:

$$\frac{du}{dx} - 2 = -\frac{(1 + 2(u - 2x) + 4x)}{1 + (u - 2x) + 2x}$$

$$= -\frac{1 + 2u}{1 + u}$$

$$\frac{du}{dx} = -\frac{1 + 2u}{1 + u} + 2$$

$$= -\frac{1 + 2u - 2(1 + u)}{1 + u}$$

$$= \frac{1}{1 + u}$$

b $\int (1 + u)\, du = \int dx$

$u + \frac{1}{2}u^2 = x + c$

$2u + u^2 = 2x + 2c$

Since $u = y + 2x$

$2(y + 2x) + (y + 2x)^2 = 2x + 2c$

$2y + 2x + y^2 + 4xy + 4x^2 = 2c$

$4x^2 + 4xy + y^2 + 2y + 2x = k$ where $k = 2c$

Challenge

The original equation is:

$$x^2\frac{dy}{dx}-xy=y^2$$

$$\frac{dy}{dx}-\frac{y}{x}=\frac{y^2}{x^2}$$

Let $v=\frac{y}{x}$

$$\frac{dv}{dx}=-yx^{-2}+x^{-1}\frac{dy}{dx}$$

$$=-\frac{y}{x^2}+\frac{1}{x}\frac{dy}{dx}$$

$$\frac{dv}{dx}+\frac{y}{x^2}=\frac{1}{x}\frac{dy}{dx}$$

$$\frac{dy}{dx}=x\frac{dv}{dx}+\frac{y}{x}$$

$$=x\frac{dv}{dx}+v$$

The original equation is:

$$x^2\frac{dy}{dx}-xy=y^2$$

The transformed equation is:

$$x\frac{dv}{dx}+v-v=v^2$$

$$x\frac{dv}{dx}=v^2$$

$$\frac{1}{v^2}\frac{dv}{dx}=\frac{1}{x}$$

$$\int\frac{1}{v^2}\,dv=\int\frac{1}{x}\,dx$$

$$-\frac{1}{v}=\ln x+c$$

Since $v=\frac{y}{x}$

$$-\frac{x}{y}=\ln x+c$$

$$y=-\frac{x}{\ln x+c}$$